Chapter 3
The Binomial Theorem

Abstract In this chapter we will introduce the Binomial Theorem, which appears in SMR 1.2 of the CSET subset 1 exam. We will also discuss different types of problems that are associated with the Binomial Theorem.

3.1 What is the Binomial Theorem?

In chapter 1 we discussed how to expand expressions like $(1+x)^2$ readily as $(1+x)^2 = 1+2x+x^2$. You can go through the same process to expand the expression if the power is 3. Expanding $(1+x)^3$ you should be able obtain the expression

$$(1+x)^3 = 1+3x+3x^2+x^3$$

How about expanding $(1+x)^6$? Now things are getting much more complicated. It is going be pretty arduous if you are going to do it manually. The binomial theorem is a theorem which will help you to readily expand expressions such as $(1+x)^6$. As a matter of fact, it straight away gives the expansion of $(a+b)^n$ for any natural number n. It may look scary at first but once you start using it, it becomes pretty easy.

Theorem 3.1 (The Binomial Theorem). *Let n be a natural number and $a, b \in \mathbb{R}$. Then,*

$$(a+b)^n = a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \cdots + C_{n-2}^n a^2 b^{n-2} + C_{n-1}^n a b^{n-1} + b^n$$

where the numbers (coefficients) C_k^n, sometimes called 'n choose k', can be easily calculated using the formula:

$$C_k^n = \frac{n!}{k!(n-k)!}$$

Recall that $n!$ (read n factorial) is defined to be $n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1$ for $n>0$, and $0! = 1$. For example, $3! = 3\cdot 2\cdot 1 = 6$. Also, note that $C_0^n = C_n^n = 1$, which explains why the first and last terms in the binomial expansion above are just a^n and b^n (with, seemingly no binomial coefficients).

Example 3.1. We want to expand $(1+x)^3$ using the binomial theorem.

Comparing $(1+x)^3$ with $(a+b)^n$, you will notice that $a=1$, $b=x$ and $n=3$. This gives you, using the binomial theorem,

$$\begin{aligned}(1+x)^3 &= 1^3 + C_1^3 1^2 x + C_2^3 1 \cdot x^2 + x^3 \\ &= 1 + C_1^3 x + C_2^3 x^2 + x^3\end{aligned}$$

Now let us calculate the coefficients, we get

$$C_1^3 = \frac{3!}{1!(3-1)!} = \frac{6}{2} = 3 \qquad C_2^3 = \frac{3!}{2!(3-2)!} = \frac{6}{2} = 3$$

It follows that

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

What is the Binomial Theorem?: Worked out examples

1. *Expand* $(1+x)^4$
   Answer: Since we want to expand $(1+x)^4$, then in this case $a=1$, $b=x$ and $n=4$. This yields, using the binomial theorem,

   $$\begin{aligned}(1+x)^4 &= 1^4 + C_1^4 1^3 x + C_2^4 1^2 x^2 + C_3^4 1 \cdot x^3 + x^4 \\ &= 1 + C_1^4 x + C_2^4 x^2 + C_3^4 x^3 + x^4\end{aligned}$$

   The coefficients we need are

   $$C_1^4 = \frac{4!}{1!(4-1)!} = \frac{24}{6} = 4 \qquad C_2^4 = \frac{4!}{2!(4-2)!} = \frac{24}{4} = 6 \qquad C_3^4 = \frac{4!}{(4-1)!1!} = \frac{24}{6} = 4$$

   It follows that

   $$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

2. *Expand* $(x+y)^5$
   Answer: In this case $a=x$, $b=y$ and $n=5$. So,

   $$(x+y)^5 = x^5 + C_1^5 x^4 y + C_2^5 x^3 y^2 + C_3^5 x^2 y^3 + C_4^5 x y^4 + y^5$$

   The coefficients we need are

   $$C_1^5 = \frac{5!}{1!(5-1)!} = \frac{12}{24} = 5 \qquad C_2^5 = \frac{5!}{2!(5-2)!} = \frac{120}{12} = 10$$

   $$C_3^5 = \frac{5!}{(5-3)!3!} = \frac{120}{12} = 10 \qquad C_4^5 = \frac{5!}{(5-1)!1!} = \frac{12}{24} = 5$$

   It follows that

   $$(x+y)^5 = x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$$

3. *Expand* $(2x-y)^4$
Answer: Since we want to expand $(2x-y)^4$ and what we want to use is $(a+b)^n$ then we get $a=2x$, $b=-y$ and $n=4$. Note that we can use the coefficients found in example 1 above. Thus, the binomial theorem yields,

$$\begin{aligned}(2x-y)^4 &= (2x)^4+C_1^4(2x)^3(-y)+C_2^4(2x)^2(-y)^2+C_3^4(2x)(-y)^3+(-y)^4\\ &= (2x)^4+4(2x)^3(-y)+6(2x)^2(-y)^2+4(2x)(-y)^3+(-y)^4\end{aligned}$$

Now we use that $(2x)^k=2^kx^k$ and that $(-y)^k=(-1)^kx^k$ to simplify what we have. We get,

$$\begin{aligned}(2x-y)^4 &= 2^4x^4+4\cdot 2^3x^3(-1)y+6\cdot 2^2x^2(-1)^2y^2+4\cdot 2x(-1)^3y^3+(-1)^4y^4\\ &= 2^4x^4-4\cdot 2^3x^3y+6\cdot 2^2x^2y^2-4\cdot 2xy^3+y^4\\ &= 16x^4-32x^3y+24x^2y^2-8xy^3+y^4\end{aligned}$$

4. *Find the first three terms of the expansion of* $\left(x-\frac{3}{x}\right)^7$
Answer: Since

$$\left(x-\frac{3}{x}\right)^7 = x^7+C_1^7x^6\left(-\frac{3}{x}\right)+C_2^7x^5\left(-\frac{3}{x}\right)^2+C_3^7x^4\left(-\frac{3}{x}\right)^3+\cdots+\left(-\frac{3}{x}\right)^7$$

then the first three terms of the expansion are

$$x^7 \qquad C_1^7x^6\left(-\frac{3}{x}\right)=-3C_1^7x^5 \qquad C_2^7x^5\left(-\frac{3}{x}\right)^2=9C_2^7x^3$$

Now, since

$$C_1^7=\frac{7!}{1!(7-1)!}=7 \qquad C_2^7=\frac{7!}{2!(7-2)!}=21$$

then the first three terms of the expansion are

$$T_1=x^7 \qquad T_2=-21x^5 \qquad T_3=189x^3$$

3.2 Binomial Expansions Using Pascal's Triangle

Another way of finding the binomial coefficients of an expansion is by using Pascal's triangle. Pascal's triangle is the following collection of rows of integers.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{array}$$

The construction of this triangle is simple. Notice that a particular number in any row, other than the first row, can be obtained by adding the two numbers in the previous row that appear above and immediately to the left and the right of the number concerned. Moreover, the coefficients in the $(k+1)^{st}$ row are exactly the binomial coefficients needed in the binomial theorem to expand $(a+b)^k$.

Example 3.2. We want to use Pascal's triangle to find the expansion of $(x+y)^5$.
Notice first that we are considering the 5^{th} power of a binomial. So, we simply go to the 6^{th} row of Pascal's triangle to look for the needed coefficients.
The entries of the 6^{th} row are: $1, 5, 10, 10, 5, 1$, and thus, setting $a = 2x$ and $b = -3y$ we get.

$$\begin{aligned}
(x+y)^5 &= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}$$

which coincides with what was found before in example 2.

3.3 Working With Specific Terms of a Binomial Expansion

We are interested in discussing the following problems:
(a) Find the 4^{th} term of a binomial expansion
(b) Find the middle term of a binomial expansion
(c) Find the coefficient with x^7 in a binomial expansion
(d) Find the term in a binomial expansion that does not contain x.

To answer questions such as these, it is a waste of time to use the full binomial expansion because we do not need to know *all* the terms in the expansion to answer a question about a single one. So, we want to get an expression that would give us a single specific term.
We notice that, in the binomial expansion

$$(a+b)^n = a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \cdots + C_{n-2}^n a^2 b^{n-2} + C_{n-1}^n ab^{n-1} + b^n$$

the first term is $T_1 = a^n$, which can be rewritten as $T_1 = C_0^n a^n b^0$.
The 2^{nd} term is $T_2 = C_1^n a^{n-1}b$, which is re-written as $T_2 = C_1^n a^{n-1} b^1$.

The third term is $T_3 = C_2^n a^{n-2} b^2$.
The fourth term is $T_4 = C_3^n a^{n-3} b^3$, etc.
It follows that the r^{th} term of the binomial expansion of $(a+b)^n$ is given by

$$T_r = C_{r-1}^n a^{n-(r-1)} b^{r-1}$$

Now we are ready to answer questions that deal with single terms in a binomial expansion.

Working With Specific Terms of a Binomial Expansion: Worked out examples

1. *Find the* 4^th *term in the expansion of* $(x-y)^7$
   Answer: In order to use the expression above for the r^{th} term of a binomial expansion we need to identify what a, b, and n are. In this case we get $a = x$, $b = -y$, and $n = 7$, and of course $r = 4$. Hence the 4^{th} term in the expansion of $(x-y)^7$ is

$$T_4 = C_3^7 x^{7-3}(-y)^3 = -C_3^7 x^4 y^3 = -35x^4y^3$$

   where in the last step we used

$$C_3^7 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} == \frac{5\cdot 6\cdot 7}{3!} = 35$$

2. *Find the coefficient with* x^3 *in the expansion of* $\left(x+\frac{1}{x}\right)^7$.
   Answer: Note that we do not know which term in the expansion is the one we need to focus on. Hence, we will need to look at the general term in the expansion of $\left(x+\frac{1}{x}\right)^7$.
   Since we know $n = 7$, $a = x$ and $b = \frac{1}{x}$ then the r^{th} term is

$$T_r = C_{r-1}^7 x^{7-(r-1)} \left(\frac{1}{x}\right)^{r-1} = C_{r-1}^7 x^{7-(r-1)} x^{-(r-1)} = C_{r-1}^7 x^{7-2(r-1)} = C_{r-1}^7 x^{9-2r}$$

   Since we want the term with x^3 then we want $x^{9-2r} = x^3$, which yields $9-2r = 3$, and thus $3 = r$. It follows that the term with x^3 is $T_3 = C_2^7 x^3$, and thus the coefficient we were looking for is $C_2^7 = 21$.

3. *Find the term with no x in the expansion of* $\left(3x-\frac{y}{x}\right)^6$.
   Answer: This problem is quite similar to the previous one, for in the previous problem you found the term that had x^3 and now you want to find the term with no x which, in other words, is the term with x^0.
   We look at the r^{th} term in the expansion of $\left(3x-\frac{y}{x}\right)^6$. Since we know $n = 6$, $a = 3x$ and $b = -\frac{y}{x}$ then we get

$$\begin{aligned} T_r &= C^6_{r-1}(3x)^{6-(r-1)}\left(-\frac{y}{x}\right)^{r-1} \\ &= C^6_{r-1}3^{6-(r-1)}x^{6-(r-1)}(-y)^{r-1}x^{-(r-1)} \\ &= C^6_{r-1}3^{6-(r-1)}(-y)^{r-1}\,x^{6-2(r-1)} \end{aligned}$$

Since we want the term with x^0 then we want $x^{6-2(r-1)} = x^0$, which yields $8-2r=0$, and thus $r=4$. It follows that the term we are looking for is

$$T_4 = C^6_3 3^3(-y)^3 = \frac{6!}{3!3!}3^3(-y)^3 = 20 3^3(-y)^3 = -540y^3$$

4. *Find the middle term of* $(3x+2y)^{40}$.
Answer: Since the expansion of $(3x+2y)^{40}$ has 41 terms, then a middle term exists, and it is exactly the 21^{st} term. Hence, the problem can be re-phrased as 'find the 21^{st} term of the expansion of $(3x+2y)^{40}$'.
Since $r=21$, $n=40$, $a=3x$ and $b=2y$ then

$$\begin{aligned} T_{21} &= C^{40}_{20}(3x)^{40-(20)}(2y)^{20} \\ &= C^{40}_{20}3^{20}x^{20}2^{20}y^{20} \\ &= C^{40}_{20}3^{20}2^{20}\,x^{20}y^{20} \end{aligned}$$

We note that the numbers in the term above are all huge, so we will leave the answer as it is now.

5. *Find a polynomial with integral coefficients which has* $1+\sqrt[4]{2}$ *as a root.*
Answer: This problem does not seem to be directly related to the binomial theorem. However, you will need the binomial expansion to solve it
Let $\alpha = 1+\sqrt[4]{2}$, and thus $\alpha - 1 = \sqrt[4]{2}$. In order to get rid of the radical we raise both sides to the 4^{th} power and get $(\alpha-1)^4 = 2$.
Now it is clear how the binomial theorem is relevant to this problem, as we use it to get

$$2 = (\alpha-1)^4 = \alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 1$$

which implies $\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha - 1 = 0$. It follows that $\alpha = 1+\sqrt[4]{2}$ is a root of $p(x) = x^4 - 4x^3 + 6x^2 - 4x - 1$.

Exercises

3.1. Evaluate
(i) $2!5!$ (ii) $4!0!$ (iii) C^5_3 (iv) C^{50}_{48}

3.2. Expand
(i) $(1+x)^5$ (ii) $(1+3y)^4$ (iii) $(a+2b)^3$ (iv) $(2x-y)^5$

3.3. Use Pascal's triangle to expand and simplify
(i) $(a+b)^6$ (ii) $(3x-5y)^4$

3.4. Find the first four terms in the expansion of $(2-3x)^8$.

3.5. Find the last three terms in the expansion of $(x-2y^2)^{12}$.

3.6. Find the coefficient of x^4 in the expansion of $(x+2y)^{10}$.

3.7. Find the coefficient of x^8 in the expansion of $\left(x^2-\frac{1}{x}\right)^7$.

3.8. Find the coefficient of the term with x^2y^2 in the binomial expansion of $(x+2y)^4$.

3.9. Find the sixth term of the binomial expansion of $(a+3b)^9$.

3.10. Find the fourth term of the binomial expansion of $\left(2-\frac{b}{2}\right)^{10}$.

3.11. Find the seventh term of the binomial expansion of $(x-y)^{15}$.

3.12. Find the fifteenth term of the binomial expansion of $(2x-1)^{17}$.

3.13. Find the middle term in the binomial expansion of
(i) $\left(x^2-\frac{1}{x}\right)^{12}$ (ii) $\left(x^2+\frac{1}{x}\right)^8$

3.14. Find the term that does not have an x in in the binomial expansion of
(i) $\left(2x+\frac{1}{x}\right)^6$ (ii) $\left(2x^2+\frac{1}{x}\right)^9$ (iii) $\left(2x^4-\frac{1}{x^2}\right)^9$

3.15. Use the binomial theorem to factor
(i) $x^5+10x^4+40x^3+80x^2+80x+32$ (ii) $8x^3+12x^2+6x+1$

3.16. Find a polynomial with integral coefficients that has $1-\sqrt[4]{3}$ as a root.

3.17. Find a polynomial with integral coefficients that has $\sqrt{2}-\sqrt[3]{3}$ as a root.